EB distributions as alternative lifetime's distributions

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***Abstract*: In this paper it was introduced a new lifetime distribution obtained as a distribution of random variable (r.v.) $\max(W_1, W_2, ..., W_K)$, where $(W_i)_{i\geq 1}$ are independent, identically exponentially distributed r.v. In the conditions of the Poisson's Limit Theorem it is shown that this distribution, called EB-Max distribution may be approximated by its analogous called EP-Max lifetime distribution and EB-Min distribution introduced in [1] may be approximated by its analogous EP-Min lifetime distribution introduced in [2].**

***Key words*: lifetime distribution, mixing r.v., exponential, zero truncated binomial and Poisson distributions, Poisson's Limit Theorem,.**

1. Introduction

In the paper [1] it was introduced EB-Min distribution compounding exponentially distributed lifetimes with zero truncated binomially distributed as r.v. as alternative to the EP-Min lifetime distribution introduced in [2] by mixing the same lifetime with zero truncated Poisson distributed r.v. in the both cases lifetimes are represented as minimum of k independent identically exponentially distributed r.v., $k = 1, 2, ...$. Our interest is to know how look their distributions if we substitute minimum by maximum and to study possible connections between them.

2. Distribution of r.v. $\max(W_1, W_2, ..., W_K)$ for random K

First of all, let us deduce a general formula for distribution of r.v. $\max(W_1, W_2, ..., W_K)$, where $(W_i)_{i\geq 1}$ are independent identically distributed random variables (i.i.d.r.v.) and K is a discrete r.v. such that $\mathbf{P}(K \in \{1, 2, ...\}) = 1$. So, we consider that distribution function (d.f.) of r.v. W_i is $F(x) = \mathbf{P}(W_i \leq x)$, $i \geq 1$. Then, due of independence of r.v. $(W_i)_{i\geq 1}$,the d.f. of r.v. $Y_k = \max(W_1, W_2, ..., W_k)$ is

$$F_{Y_k}(x) = \mathbf{P}(Y_k \leq x) = \mathbf{P}(\max(W_1, W_2, ..., W_k) \leq x) =$$
$$= \mathbf{P}(W_1 \leq x, W_2 \leq x, ..., W_k \leq x)$$
$$= [F(x)]^k, \ \forall k = 1, 2, ...$$

This means that d.f. of r.v. $Y = \max(W_1, W_2, ..., W_K)$ is a mixture of d.f. $F_{Y_k}(x)$ with respect to the distribution of r.v. K. Indeed,

$$F_Y(x) = \mathbf{P}(Y \leq x) = \mathbf{P}(\max(W_1, W_2, ..., W_K) \leq x) =$$
$$\sum_{k\geq 1} \mathbf{P}(\max(W_1, W_2, ..., W_k) \leq x)\mathbf{P}(K = k) =$$
$$\sum_{k\geq 1} [F(x)]^k \mathbf{P}(K = k) \quad (1)$$

This formula show us that, if $(W_i)_{i\geq 1}$ are r.v. of absolutely continuous type, then Y is a r.v. of the same type and its probability density function (p.d.f.) is

$$f_Y(x) = F_Y'(x) = \sum_{k\geq 1} kF'(x)[F(x)]^{k-1}\mathbf{P}(K = k) \quad (2)$$

3. EB-Max distribution

Now we may apply formulas (1)-(2) to introduce a new lifetime distribution called EB-Max distribution.

Proposition 1. *If* $(W_i)_{i\geq 1}$ *are independent identically exponentially distributed r.v. with parameter* λ , $\lambda > 0$, i.e.,

$$F(x) = \mathbf{P}(W_i \leq x) = (1 - e^{-\lambda x}) \cdot I_{[0,+\infty)}(x),\ i \geq 1$$

and K *is a zero truncated binomially distributed r.v., i.e.,*

$$\mathbf{P}(K = k) = \frac{1}{1-(1-p)^n} C_n^k\, p^k (1-p)^{n-k},$$

$$k = \overline{1,n},\ \ p \in (0,1),$$

then d.f. of r.v. $Y = \max(W_1, W_2, ..., W_K)$, *is given by formula*

$$U_{\max}(x) = \frac{(1-pe^{-\lambda x})^n - (1-p)^n}{1-(1-p)^n} \cdot I_{[0,+\infty)}(x) \quad (3)$$

and p.d.f. of r.v. Y *is*

$$u_{\max}(x) = \frac{np\lambda e^{-\lambda x}}{1-(1-p)^n}(1-pe^{-\lambda x})^n \cdot I_{[0,+\infty)}(x) \quad (4)$$

where

$$I_{[0,+\infty)}(x) = \begin{cases} 0, \text{ if } x < 0, \\ 1, \text{ if } x \geq 0. \end{cases}$$

Proof. From (1) we have that d.f.

$$U_{\max}(x) = \mathbf{P}(Y \leq x) = \sum_{k=1}^{n} \frac{1}{1-(1-p)^n} C_n^k\, p^k (1-p)^{n-k} (1-e^{-\lambda x})^k$$

$$\times I_{[0,+\infty)}(x) = \frac{(1-pe^{-\lambda x})^n - (1-p)^n}{1-(1-p)^n} \cdot I_{[0,+\infty)}(x).$$

So, p.d.f.

$$u_{\max}(x) = U'_{\max}(x) = \frac{np\lambda e^{-\lambda x}}{1-(1-p)^n}(1-pe^{-\lambda x})^{n-1} \times$$

$$\times I_{[0,+\infty)}(x).$$

□

Corollary. *If* Y *is the EB-Max distributed r.v. then*

a) *median value of* Y *is*

$$y_m = -\frac{1}{\lambda}\ln\left(\frac{\sqrt[n]{\frac{1}{2}(1+(1-p)^n}}{p}\right);$$

b) *for each* $r = 1, 2, ...$ *the* r *-moment of r.v.* Y *is*

$$EY^r = \frac{r!}{\lambda^r\left[1-(1-p)^n\right]}\sum_{i=0}^{n-1}(-1)^i C_{n-1}^i \frac{p^{n-i}}{(n-i)^{r+1}};$$

c) *mean value of* Y *is*

$$EY = \frac{1}{\lambda\left[1-(1-p)^n\right]}\sum_{i=0}^{n-1}(-1)^i C_{n-1}^i \frac{p^{n-i}}{(n-i)^2};$$

d) *variance of* Y *is*

$$DY = \frac{2}{\lambda^2\left[1-(1-p)^n\right]}\sum_{i=0}^{n-1}(-1)^i C_{n-1}^i \frac{p^{n-i}}{(n-i)^3} -$$

$$-\frac{1}{\lambda^2\left[1-(1-p)^n\right]^2}\left(\sum_{i=0}^{n-1}(-1)^i C_{n-1}^i \frac{p^{n-i}}{(n-i)^2}\right)^2;$$

e) *survival function is*

$$s(x) = 1 - U_{\max}(x) = \frac{1-(1-pe^{-\lambda x})^n}{1-(1-p)^n},\ x \geq 0;$$

f) *hazard function is given by*

$$h(x) = \frac{u_{\max}(x)}{s(x)} = \frac{np\lambda e^{-\lambda x}(1-pe^{-\lambda x})^{n-1}}{1-(1-pe^{-\lambda x})^n},\ x \geq 0.$$

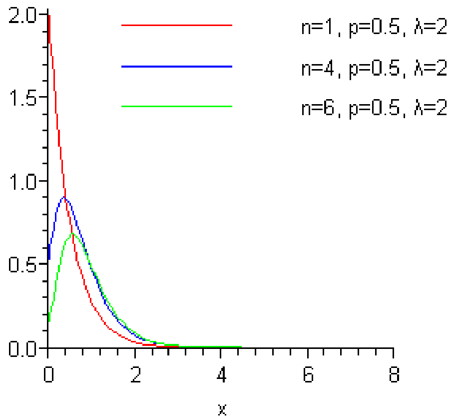

Figure.1. The p.d.f. of EB-Max distribution for different values of parameters.

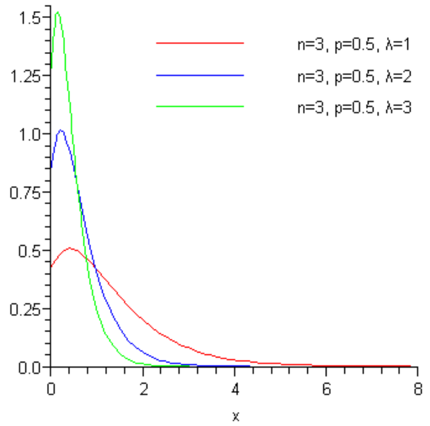

Figure 2. The p.d.f. of EB-Max distribution for different values of parameters.

4. Approximating EB distributions by EP distributions

For the same lifetimes $(W_i)_{i\geq 1}$, substituting zero truncated binomial distribution for r.v. K by zero truncated Poisson distribution with parameter μ, $\mu > 0$, i.e.,

$$\mathbf{P}(K=k) = \frac{1}{1-e^{-\mu}}\frac{\mu^k}{k!}e^{-\mu}, k = 1, 2, ...,$$

authors of the work [3] was introduced another new lifetime distribution called EP-Max distribution given by formula

$$V_{\max}(x) = \frac{e^{-\mu e^{-\lambda x}} - e^{-\mu}}{1-e^{-\mu}} \cdot I_{[0,+\infty)}(x). \qquad (5)$$

In the similar ways in [1]-[2] it was introduced EB-Min and EP-Min life time distributions. According to the [1] EB-Min distribution is given by d.f.

$$U_{\min}(x) = \{1 - \frac{1}{1-(1-p)^n}\{[1 - $$

$$-(1-p)^n\}\} I_{[0,+\infty)}(x)$$

and according to the [2] EP-Min distribution is given by d.f.

$$V_{\min}(x) = \frac{e^{\mu e^{-\lambda x}} - e^{\mu}}{1-e^{\mu}} \cdot I_{[0,+\infty)}(x). \qquad (7)$$

As we know, Poisson's Limit Theorem [4] show us that, in some conditions, binomial distribution may by approximated by Poisson distribution. This fact suggest us that between d.f. $U_{\max}(x)$ and $V_{\max}(x)$ and, on the other hand, between d.f. $U_{\min}(x)$ and $V_{\min}(x)$ does exist the similar connections. Indeed, it is true the following

Proposition (Poisson's Limit Theorem for EB an EP distributions). *In the conditions of the Poisson's Limit Theorem, i.e., if* $n \to +\infty$ *and* $p \to 0$ *in such way that* $np \to \mu$, $\mu > 0$, *then*

$$\lim_{\substack{n\to+\infty \\ p\to 0}} U_{\max}(x) =$$

$$= \lim_{\substack{n\to+\infty \\ p\to 0}} \frac{(1-pe^{-\lambda x})^n - (1-p)^n}{1-(1-p)^n} \cdot I_{[0,+\infty)}(x)$$

$$= V_{\max}(x), \ \forall x \in \mathrm{R}$$

and

$$\lim_{\substack{n\to+\infty \\ p\to 0}} U_{\min}(x) =$$

$$= \lim_{\substack{n\to+\infty \\ p\to 0}} \left\{1 - \frac{1}{1-(1-p)^n}\left\{[1 - p(1-e^{-\lambda x})]^n - (1-p)^n\right\}\right\} \cdot$$

$$\cdot I_{[0,+\infty)}(x) =$$

$$= V_{\min}(x), \ \forall x \in \mathrm{R}.$$

Proof. Let us observe that in the conditions $n \to +\infty$ and $p \to 0$ in such way that $np \to \mu$, $\mu > 0$, in fact our Proposition is a consequence of the following equalities:

$$\lim_{\substack{n\to+\infty \\ p\to 0}} (1-p)^n = \lim_{\substack{n\to+\infty \\ p\to 0}} \left(1 - \frac{\mu}{n} + o(\frac{\mu}{n})\right)^n = e^{-\mu},$$

$$\lim_{\substack{n\to+\infty \\ p\to 0}} (1-pe^{-\lambda x})^n = \lim_{\substack{n\to+\infty \\ p\to 0}} \left(1 - \frac{\mu}{n}e^{-\lambda x} + o(\frac{\mu}{n})\right)^n = e^{-\mu e^{-\lambda x}},$$

and

$$\lim_{\substack{n\to+\infty \\ p\to 0}} [1 - p(1-e^{-\lambda x})]^n = \lim_{\substack{n\to+\infty \\ p\to 0}} \left[1 - \frac{\mu}{n}(1-e^{-\lambda x}) + o(\frac{\mu}{n})\right]^n$$

$$= e^{-\mu(1-e^{-\lambda x})}. \quad \square$$

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