STABILITY OF ELASTIC BODIES WITH MICROSTRUCTURE

M. Ispas, D. Manole
Technical University „Gh.Asachi" of Iassy

INTRODUCTION

Stability of solutions in linear elasticity has been considered in [3]. Sufficient stability conditions for the solution of linear dynamic viscoelasticity and in linear dynamic micropolar viscoelasticity are presented in [4] and [5].

We are dealing here with the stability of the equilibrium solution of homogeneous mixed initial boundary-value problem.

1. PRELIMINARY CONSIDERATIONS

Throughout this paper, we employ a rectangular coordinate system x_K and the indicial notation. Consider an elastic medium with microstructure occupying the domain (Ω) of the three-dimensional. Euclidian space, whose boundary is $(\partial\Omega)$, in the time $[0, T]$, $0<T<\infty$. The basic equations in the linear theory of these bodies are [1]:

- the equations of motion:

$$\begin{cases} \left(\tau_{ij}+\sigma_{ji}\right)_{,j}+F_i=\rho\cdot \ddot{u}_i \\ \tau_{kij,k}+\sigma_{ij}+L_{ij}=I_{is}\cdot \ddot{\Psi}_{sj}\,;\,\tau_{ij}=\sigma_{ji} \end{cases} \quad (1)$$

in $\Omega x]0, T[$, for any fixed T;

- the constitutive law:

$$\begin{cases} \tau_{ij}=a_{ijkl}\cdot\varepsilon_{ke}+g_{klij}\cdot\gamma_{ke}+f_{kmnij}\cdot\chi_{kmn}\ , \\ \sigma_{ij}=g_{ijkl}\cdot\varepsilon_{ke}+b_{klij}\cdot\gamma_{kl}+d_{ijkmn}\cdot\chi_{kmn}\ , \\ \mu_{ijk}=f_{ijkmn}\cdot\varepsilon_{mn}+d_{mnijk}\cdot\gamma_{mn}+c_{ijkmen}\cdot\chi_{kme}\ , \end{cases} \quad (2)$$

- the cinematic relations:

$$\begin{cases} \varepsilon_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right)\ , \\ \gamma_{ij}=u_{j,i}+\psi_{ij}\ , \\ \chi_{ijk}=\psi_{jk,i}\ . \end{cases} \quad (3)$$

In the above equations, we have used the following notation: u_i – components of the displacement vector; ψ_{ij} – components of the microdisplacement tensor; ε_{ij}, γ_{ij}, χ_{ijk} – kinematic characteristics of the strain; F_i – components of the body force; L_{ij} – components of body microforce; τ_{ij} – components of the classical stress tensor; σ_{ij} – components of the relative stress tensor; τ_{ijk} – components of the couple-stress tensor; $\rho(x)$, $I_{ij}(x)$, $a_{ijlk}(x)$, $b_{ijkl}(x)$, $c_{ijkmnl}(x)$, $g_{ijkl}(x)$, $f_{ijkmn}(x)$, $d_{ijkmn}(x)$, characteristic functions of the material, the comma denotes partial differentiations with respect to the space variables xi, a dot denotes partial derivation with respect to time.
We assume that the characteristic functions of the material are bounded and measurable functions in $(\overline{\Omega})=(\Omega)\cup(\partial\Omega)$, and satisfy:

$$\begin{cases} \rho(x)\geq\ \rho_0\rangle\ 0 \\ i_{ij}(x)=i_{ji}(x)\,,\,I_{jk}\cdot\xi_{ij}\cdot\xi_{ik}\geq I\cdot\xi_{ij}\cdot\xi_{ji} \end{cases} \quad (4)\ ,$$

for any tensor $\xi(\xi_{ij})$, i - being a constant > 0, and:

$$\begin{cases} a_{ijkl}=a_{klij}=a_{jikl}\ ,\ b_{lijk}=b_{klij} \\ c_{ijkmnl}=c_{mnlijk}\ , \\ f_{ijkmn}=f_{jikmn}\ ,\ g_{ijkl}=g_{ijlk}\ . \end{cases} \quad (5)$$

To the system of field equations, we add the initial conditions:

$$\begin{cases} u_i(x,b)=a_i(x)\,,\,\dot{u}_i(x,0)=b_i(x) \\ \Psi_{ij}(x,0)=c_{ij}(x)\,,\,\dot{\Psi}_{ij}(x,0)=d_{ij}(x)\ , \end{cases} \quad (6)$$

$x\in(\Omega)$ and the homogeneous boundary conditions:

$$\begin{cases} u_i(x,t)=0\,,\,on\,(\partial\Omega_u)\cup]0,T[\ , \\ t_i(x,t)=(\tau_{ji}+\sigma_{ji})n_j=0,on\,(\partial\Omega_t)X]0,T[\ , \\ \Psi_{ij}(x,t)=0\,,\,on\,(\partial\Omega_\Psi)\cup]0,T[\ , \\ T_{ij}(x,t)=\mu_{kji}n_k=0\,,on\,(\partial\Omega_T)X]0,T[\ , \end{cases} \quad (7)$$

where $\mathbf{a_i}$, $\mathbf{b_i}$, $\mathbf{c_{ij}}$, $\mathbf{d_{ij}}$ are prescribed functions, $\mathbf{n_i}$ are components of the unit outward normal to $(\partial\Omega)$ and $(\partial\Omega_u),(\partial\Omega_t),(\partial\Omega_\Psi),(\partial\Omega_T)$ denote subset of such that:

$$\begin{cases}(\partial\Omega)=(\partial\Omega_u)\cup(\partial\Omega_t)=(\partial\Omega_\Psi)\cup(\partial\Omega_T)\ ; \\ (\partial\Omega_u)\cap(\partial\Omega_T)=(\partial\Omega_\Psi)\cap(\partial\Omega_T)=\Phi\end{cases}$$

Let $C_0^\infty(\Omega)$ be the vector functions with compact support in (Ω) and components of $C^\infty(\Omega)$.

Let $\mathbf{H_0}$, $\mathbf{H_+}$ be the Hilbert spaces obtained by completion of $C_0^\infty(\Omega)$ under the norms $\|\cdot\|_0$, $\|\cdot\|_+$ induced by inner products

$$(u,v)_{H_0}=\int_\Omega(u_i\cdot v_i+\psi_{jk}\cdot\varphi_{jk})d\Omega\ ,$$
$$(u,v)_{H_+}=\int_\Omega(u_{i,j}\cdot v_{i,j}+\psi_{ij,k}\cdot\varphi_{ij,k})d\Omega\ ,$$

respectively, and let $\mathbf{H_-}$ be the completion of $C_0^\infty(\Omega)$ by means of the norm:

$$\|u\|_-=\sup_{v\in H_+}\frac{|(u,v)_{H_0}|}{\|v\|_+}\ ,\ where\ u=(u_i,\psi_{jk})\ ,$$
$$v=(v_i,\varphi_{jk})\ .$$

We introduce the notation:

$$A(\varepsilon_{ij},\gamma_{ij},\chi_{ijk})=a_{ijkl}\cdot\varepsilon_{ij}\cdot\varepsilon_{kl}+b_{ijkl}\cdot\gamma_{ij}\cdot\gamma_{kl}+$$
$$+c_{ijkmne}\cdot\chi_{ijk}\cdot\chi_{mnl}+2\cdot g_{ijkl}\cdot\gamma_{ij}\cdot\varepsilon_{kl}+$$
$$+2\cdot f_{ijkmn}\cdot\chi_{ijk}\cdot\varepsilon_{mn}+2\cdot d_{ijkmn}\cdot\gamma_{ij}\cdot\chi_{kmn}\ ,\qquad (8)$$

$$E(t)=\frac{1}{2}\int_\Omega(\rho\cdot\dot u_i^2+I_{is}\cdot\dot\psi_{sj}\cdot\dot\psi_{ij}+A(\varepsilon_{ij},\gamma_{ij},\chi_{ijk}))d\Omega\qquad (9)$$

$$p(t)=\int_\Omega(F_i\cdot\dot u_i+L_{ij}\cdot\dot\psi_{ij})d\Omega\qquad (10)$$

We suppose that:

$$\int_\Omega A(\varepsilon_{ij},\gamma_{ij},\chi_{ijk})d\Omega\geq\alpha\cdot\|u(x,t)\|_+^2\ ,\qquad (11)$$
$$\alpha=const.\geq 0$$

2. STABILITY ANALYSIS

The null solution is stable under perturbation u_i,ψ_{ij} satisfying (1) – (7) if for any $\varepsilon>0$ there exists a δ_ε such that [3]:

$$\left[\int_{\Omega(0)}(\rho\cdot u_i\cdot u_i+I_{ij}\cdot\psi_{is}\cdot\psi_{js})d\Omega+Q\right]<\delta\qquad (12)$$

implies that:

$$\sup_{0\leq t<T}\left[\int_{\Omega(t)}(\rho\cdot u_i\cdot u_i+I_{ij}\cdot\psi_{is}\cdot\psi_{js})d\Omega+Q\right]<\varepsilon\qquad (13)$$

where $\Omega(t)$ denotes integration over the volume of the body at time t, while Q is an appropriately chosen positive functional of the initial data which tends to zero as the initial data tend to zero. Its precise form will be specified later. We say that a solution is unstable when it is not stable.

Theorem 2.1. In condition (11) the null solution is stable for $\mathbf{F_i=0}$, $\mathbf{L_{ij}=0}$.
Proof. Consider the functions G(t) defined by:

$$G(t)=\ln[F(t)+Q]+t^2\qquad (14)$$

where:

$$F(t)=\int_{\Omega(t)}(\rho\cdot u_i\cdot u_i+I_{ij}\cdot\psi_{is}\cdot\psi_{js})d\Omega\qquad (15)$$

We have:

$$(F+Q)^2\cdot\ddot G\equiv(F+Q)\cdot\ddot F-(\dot F)^2+2(F+Q)^2\geq 0,$$
$$0\leq t\leq T\qquad (16)$$

From (15) and (4) we obtain:

$$\dot F=2\int_{\Omega(t)}(\rho\cdot u_i\cdot\dot u_i+I_{ij}\cdot\psi_{is}\cdot\dot\psi_{js})d\Omega\qquad (17)$$

and:

$$\ddot F=2\int_{\Omega(t)}(\rho\cdot\dot u_i\cdot\dot u_i+I_{ij}\cdot\dot\psi_{is}\cdot\dot\psi_{js}+\rho\cdot u_i\cdot\ddot u_i+$$
$$+I_{ij}\cdot\psi_{is}\cdot\ddot\psi_{js})d\Omega\qquad (18)$$

Applying the divergence theorem and taking into account (1), (7), from (18), we have:

$$\ddot{F} = 2 \int_{\Omega(t)} [(\rho \cdot \dot{u}_i \cdot \dot{u}_i + I_{ij} \cdot \dot{\psi}_{is} \cdot \dot{\psi}_{js} - (\tau_{ij} \cdot \varepsilon_{ij} +$$

$$+\sigma_{ij} \cdot \gamma_{ij} + \mu_{ijk} \cdot \chi_{ijk})] d\Omega \qquad (19)$$

From (19) and (9), we get:

$$\ddot{F} = -4E(t) + 4 \int_{\Omega(t)} (\rho \cdot \dot{u}_i \cdot \dot{u}_i + I_{ij} \cdot \dot{\psi}_{is} \cdot \dot{\psi}_{js}) d\Omega \quad (20)$$

Since E(t) defined by (9) is time-independent (i.e. E(0) = E(t)), from (19) we obtain:

$$\ddot{F} = -4E(0) + 4 \int_{\Omega(t)} (\rho \cdot \dot{u}_i \cdot \dot{u}_i + I_{ij} \cdot \dot{\psi}_{is} \cdot \dot{\psi}_{js}) d\Omega \quad (21)$$

Taking into account (21), (17) we use Schwarz's inequality to obtain:

$$(F+Q)\ddot{F} - (\dot{F})^2 \geq -4E_{(0)}(F+Q) +$$

$$+4Q \int_{\Omega(t)} \rho \dot{u}_i^2 \, d\Omega \geq 4E_{(0)}(F+Q) \geq -2Q(F+Q) \geq$$

$$\geq -2(F+Q)^2 \quad , \qquad (22)$$

provided Q is chosen to satisfy $Q \geq 2E_{(0)}$.

Thus (16) is established.

From (16) there results the convexity on G(t) on [0, T].

From the convexity of G(t), it immediately follows that:

$$G(t) \leq G(\frac{t}{T}T + (1-\frac{t}{T}) \cdot 0) \leq \frac{t}{T}G(t) + (1-\frac{t}{T}) \cdot G(0),$$

$$0 \leq t \leq T, \qquad (23), \quad \textit{i.e.}$$

$$F(t) + Q \leq e^{t(T-t)} \cdot [F(T)+Q]^{\frac{t}{T}} \cdot [F(0)+Q]^{1-\frac{t}{T}},$$

$$0 \leq t \leq T \qquad (24)$$

Since all term on the right of (24) remain bounded, it follows that for **0≤t<T** arbitrarily small values of **F(0) + Q** imply arbitrarily small values of **F(t) + Q.**

This concludes the proof of the theorem.

Theorem 2.2. The equilibrium solution of the linear dynamic theory of elastic media with microstructure is uniformly Liapunov stable with respect to the measures

$$\mu(u) = \left\| \dot{u}(x,t) \right\|_0^2 + \left\| u(x,t) \right\|_+^2 ,$$

$$\mu_0(u) = \left\| \dot{u}(x,0) \right\|_0^2 + \left\| u(x,0) \right\|_+^2 \qquad (25)$$

or in respect to the measures:

$$\mu(u) = \left\| \dot{u}(x,t) \right\|_0^2 + \left\| u(x,t) \right\|_+^2 \cdot \mu_0(u) = E(0) \quad (26)$$

Proof. We have

$$\mathbf{E(t) = E(0)}, \text{ for } \mathbf{t > 0.} \qquad (27)$$

Using relation (4), (11), we get

$$\rho_0' \cdot \left\| \dot{u}(x,t) \right\|_0^2 + \alpha \cdot \left\| u(x,t) \right\|_+^2 \leq 2E(t), \quad (28)$$

where $\rho_0' = \min\{\rho_0, I\}$ and thus from **(27)**, we have:

$$\left\| \dot{u}(x,t) \right\|_0^2 + \alpha \cdot \left\| u(x,t) \right\|_+^2 \leq \frac{2}{m} E(t), \qquad (29)$$

where $m = \min\{\rho_0', \alpha\}$.

If we introduce the notation

$$\rho_0'' = \max\{\rho, E\}, \; N = \operatorname*{ess\,sup}_{x \in \Omega} \sqrt{tr(P^T \cdot P)} \; ,$$

$$E = \operatorname*{ess\,sup}_{x \in \Omega} \sqrt{tr(R^T \cdot R)}$$

$$M = \max(\{\rho_0'', N\}), \; R_{ijkl} = \delta_{ijkl} I_{jl} \quad ,$$

$$p = \begin{pmatrix} A_{9x9} & G_{9x9} & 0 & 0 & 0 & 0 & 0 \\ G_{9x9} & C_{9x9} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{27x27} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & F_{9x27} & 0 & 0 \\ 0 & 0 & 0 & F_{29x7} & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & D_{9x27} \\ 0 & 0 & 0 & 0 & 0 & F_{29x7} & 0 \end{pmatrix}_{117x117} ,$$

R, A, B, C, G, F, D is the matrix of components $R_{ijkl}(x)$, $A_{ijkl}(x)$, $B_{ijkl}(x)$, $C_{ijkmnl}(x)$, $G_{ijkl}(x)$, $F_{ijkmn}(x)$, $D_{ijkmn}(x)$.

We obtain:

$$2E_0 = \int_\Omega [\rho \cdot \dot{u}_i^2(x) + R_{ijkl} \cdot \dot{\psi}_{ij}(x,0) \cdot \dot{\psi}_{kl}(x,0) +$$
$$+a_{ijk} \cdot \varepsilon_{ij}(x,0) \cdot \varepsilon_{kl}(x,0) + b_{ijk} \cdot \gamma_{ij}(x,0) \cdot \gamma_{kl}(x,0) +$$
$$+c_{ijkmnl} \cdot \chi_{ijk}(x,0) \cdot \chi_{mnl}(x,0) + 2g_{ijkl} \cdot \gamma_{ij}(x,0) \cdot$$
$$\cdot \varepsilon_{kl}(x,0) + 2f_{ijkmn} \cdot \chi_{ijk}(x,0) \cdot \varepsilon_{mn}(x,0) +$$
$$+2d_{ijkmn} \cdot \gamma_{ij}(x,0) \cdot \chi_{kmn}(x,0) d\Omega \le$$

$$\le \int_\Omega [\rho_0'' \cdot ((\dot{u})_i^2(x,0) + (\dot{\psi})_{ij}^2(x,0) + N((\dot{\varepsilon})_{ij}^2(x,0) +$$
$$+(\dot{\chi})_{ijk}^2(x,0)) d\Omega \le M \int_\Omega [(\dot{u})_i^2(x,0) + (\dot{\psi})_{ij}^2(x,0)$$
$$+\varepsilon_{ij}^2(x,0) + \chi_{ijk}^2(x,0)] d\Omega \le M_1 [\left\|\dot{u}(x,0)\right\|_0^2 +$$
$$+\alpha \cdot \left\|u(x,t)\right\|_+^2] \text{ , } M_1 > 0. \qquad (30)$$

Now, from (29) and (30), we obtain the main inequality:

$$\left\|\dot{u}(x,t)\right\|_0^2 + \left\|u(x,t)\right\|_+^2 \le \frac{M_1}{m} \times [\left\|\dot{u}(x,0)\right\|_0^2 +$$
$$+\left\|u(x,0)\right\|_+^2] \qquad (31)$$

This concludes the proof of the theorem.

<u>Corollary 2.2</u>. Using Friedrichs's inequality [2, pp.41]

$$\int_\Omega \sum_{i=1}^{12} u_i^2 d\Omega \le c_i \left\{ \int_\Omega \sum_{i,k=1}^{12} u_{i,k}^2 d\Omega + \int_{\partial\Omega} \sum_{i=1}^{12} u^2 d\sigma \right\},$$
$$c_1 > 0, \qquad (32)$$

we have:

$$\left\|u(x,t)\right\|_0^2 \le c_1 \left\|u(x,t)\right\|_+^2,$$
$$\text{if } \partial\Omega_t = \partial\Omega_T = \phi \qquad (33)$$

From (31) and (33), we have

$$\left\|\dot{u}(x,t)\right\|_0^2 + \left\|u(x,t)\right\|_+^2 \le \frac{M_1}{m \cdot m_1} \cdot [\left\|\dot{u}(x,0)\right\|_0^2 +$$
$$+\left\|u(x,0)\right\|_+^2] \qquad (34)$$

with $\mathbf{m_1 = \min\{1, c_1\}}$. That is, the equilibrium solution is stable with respect to the measures

$$\mu(u) = \left\|\dot{u}(x,t)\right\|_0^2 + \left\|u(x,t)\right\|_+^2, \ \mu_0(u) = E(0) \qquad (35)$$

or in respect to the measures

$$\mu(u) = \left\|\dot{u}(x,t)\right\|_0^2 + \left\|u(x,t)\right\|_+^2, \qquad (36)$$
$$\mu_0(u) = \left\|\dot{u}(x,0)\right\|_0^2 + \left\|u(x,0)\right\|_+^2.$$

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