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RESPONCE OF HALF-SPACE TO THE DYNAMICAL ACTION OF A HEAT SOURCE

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Abstract. This article presents the new results, obtained in a an explicit form, regarding the construction of the influence functions and integral formula of Poisson type for the half-space of an important limit problem concerning the no-coupled dynamic thermoelasticity. For this purpose the author made use of a new method, elaborated and published by her in the previous works. This method allows to obtain the integral formula and Green functions for a large variety of problems concerning dynamic thermoelasticity. The main advantage of this method consists in the combination of two solution stages of the problems regarding dynamic thermoelasticity in one single stage.
Key words: Green functions, Half-space, Poisson formula, Thermoelasticity, Volume dilation.

INTRODUCTION

In the theory of uncoupled heat conduction (V. Budac, A. Samarskii, A. Tihonov, 1980; A. Kartashov, 1980), that is a constitutional part of the theory of thermal stresses, to solve (BVP) an integral Green's formula is suggested to determine the temperature field resulted from the thermal exposure. The analogous formula was suggested by Green to determine the field of elastic displacements (W. Nowacki, 1962; V. Şeremet, 2003), produced by the known mechanical actions in a case of BVP in the linear theory of elasticity. In a case of BVP in the theory of thermal stresses (uncoupled thermo elasticity) the corresponding integral formula used to determine thermo elastic displacements was obtained by

Maysel (W. Nowacki, 1962). However, there is a principal difference between Maysel's and Green's formulae that can be described as following. In the integral Green's formulae the desired values (the temperature field in the heat conduction problems or the field of displacements in the linear elasticity problems) are given directly through the prescribed initial actions (inner heat source, surface temperature, etc.-in the heat conduction problems or body forces, surface stresses, etc.- in the problems of linear elasticity). In a case of integral Maysel's formula, the desired values (the thermo elastic displacements in the thermo elasticity problems) are not represented directly in terms of prescribed initial thermal actions. They are represented in terms of temperature field, which is an intermediate unknown value in the given problem and is to be pre-determined. This fact introduces certain inconvenience in application of Maysel's formula. The thermo elastic influence functions and the general Green's integral formula in dynamical thermo elasticity have been obtained for the first time in the papers (V. Şeremet, 2003; V. Sheremet, 2003). In the present paper the above mentioned results (V. Sheremet, 2002) have been applied to the solution of a new very important BVP in dynamical thermo elasticity for a half-space.

MATERIAL AND METHOD

Dynamic Response of an Elastic Solid Body to a Unitary Point Heat Source

Generalization of influence function and Green's formulae onto the BVP in the classical theory of static and dynamical thermo elasticity (W. Nowacki, 1962) was given for the first time in the papers (V. Şeremet, 2003 ; V. Sheremet, 2003). The considered influence functions $U_k(x,\xi;t-\tau)$ have the physical means as the displacements generated by a unitary point heat source. It represents a convolution over the body volume V of two influence functions. The first function is the Green's function $G(x,z;t-s)$ for the BVP in heat conduction. These functions are the functions of influence of a unitary heat source onto the temperature. The second function is the function of the influence of unitary temperature onto the displacements for the BVP in the dynamical thermo elasticity. These functions are equivalent to the influence functions of unitary concentrated body forces onto the volume dilatation $\Theta^{(k)}(z,\xi;s-\tau)$. Thus, the introduced functions of the influence of a unitary heat source onto the thermo elastic displacements are determined by the general integral formula:

$$U_k(x,\xi;t-\tau)=\gamma\int_0^t ds\int_V G(x,z;t-s)\Theta^{(k)}(z,\xi;s-\tau)dV(z) \qquad (1)$$

In Eqs. (1): $\gamma=\alpha_t(2\mu+3\lambda)$; α_t – the coefficient of the linear thermal dilatation; λ,μ – Lame's constants of elasticity. Finally, the influence functions $U_k(x,\xi;t-\tau)$ take into consideration both physical phenomena (heat conduction as well as elasticity) in the solid body:

1) Over the coordinates of the point of observation $x\equiv(x_1,x_2,x_3)$ the thermo elastic displacements, satisfies the equations of the BVP for the Green's functions in the theory of heat conduction. The only difference is that the unitary heat source is replaced by the influence function of the unitary concentrated forces onto the bulk dilatation:

$$\nabla_x^2 U_k(x,\xi;t-\tau)=-\gamma\Theta^{(k)}(x,\xi;t-\tau). \qquad (2)$$

The initial and boundary conditions also have to be used.

2) Over the coordinates of the point of application $\xi\equiv(\xi_1,\xi_2,\xi_3)$ of the heat source, it satisfies the BVP for the Green's matrix components. The only difference is that the concentrated body force is replaced by derivatives from the Green's function in heat conduction:

$$\mu\nabla^2 U_k(x,\xi;t-\tau)+(\lambda+\mu)\Theta_{,k}(x,\xi;t-\tau)-\rho\,\partial^2 U_i(x,\xi;t-\tau)/\partial t^2-\gamma G_{,k}(x,\xi;t-\tau)=0. \qquad (3)$$

The initial and boundary conditions also have to be used. The volume dilatation has to be determined from the following Lame set of equations:

$$\nabla^2 U_i^{(k)}(x,\xi;t-\tau)+(\lambda+\mu)\Theta_{,i}^{(k)}(x,\xi;t-\tau)-\rho\,\partial^2 U_i(x,\xi;t-\tau)/\partial t^2=-\delta_{ik}\delta(x-\xi)\delta(t-\tau);\quad i,k=1,2,3 \qquad (4)$$

with respective initial and boundary conditions

Analog of the Green's Formula for an Elastic Solid Body in Dynamical Thermo Elasticity

On the base of introduced influence functions in the papers (V. Sheremet, 2002 ; V. Sheremet, 2003) were suggested the following new Green's type integral formula for displacements:

$$U_k(\xi,t)=\int_V \Phi_0(z)U_k(z,\xi;t)dV(z)+\int_0^t d\tau\int_V F(z,\tau)U_k(z,\xi;t-\tau)dV(z)-$$
$$-a\int_0^t d\tau\int_{\Gamma_D} T(y,\tau)\frac{\partial U_k(y,\xi;t-\tau)}{\partial n_y}d\Gamma_D(y)+a\int_0^t d\tau\int_{\Gamma_N}\frac{\partial T(y,\tau)}{\partial n_y}U_k(y,\xi;t-\tau)d\Gamma_N(y)+ \tag{5}$$
$$a\int_0^t d\tau\times\int_{\Gamma_M}\left[aT(y,\tau)+\alpha\frac{\partial T(y,\tau)}{\partial n_y}\right]U_k(y,\xi;t-\tau)d\Gamma_M(y)$$

In Eqs (5) is noted: $U_k(y,\xi;t-\tau),\ y\in\Gamma_N$; $\partial U_k(y,\xi;t-\tau)/\partial n_y,\ y\in\Gamma_D$ and $U_k(y,\xi;t-\tau),\ y\in\Gamma_M$ - the functions of influence of a unitary surface heat flux $\partial T(y,\tau)/\partial n_y=\delta(x-y)\delta(t-\tau)$, surface temperature $T(y,\tau)=\delta(x-y)\delta(t-\tau)$ and of a unitary low of heat exchange between exterior medium and surface of the body $aT(y,\tau)+\alpha(\partial T(y,\tau)/\partial n_y)=\delta(x-y)\delta(t-\tau)$ onto dynamical thermo elastic displacements. The formula in Eqs (1)-(3) and (5) has been obtained for the first time in (V. Sheremet, 2002) and are the generalization of the well-known Green's functions and Green's integral formula from the theories of heat conduction and elasticity onto dynamical thermo elasticity. More, Eq (5) is the generalization of Maysel's formula (W. Nowacki, 1962) onto those cases when the temperature field satisfies the equation of heat conduction, the field being caused by the internal heat source $F(x,t)$ and temperature $T(y)$ or heat flux $\partial T(y,t)/\partial n_y$ prescribed on the boundary.

REZULTS AND DISCUTIONS

Response of an Elastic Half-space to a Dynamic Unitary Point Heat Source

This section gives an example for determination of dynamical thermo elastic displacements in a form of integrals for a half-space, which are the particular case of the general integral formula in Eq (64). To do this, first, the functions of influence of the unitary heat source on the dynamical thermo elastic displacements should be constructed. So, let us to determine the displacements in the half-space $(0\le x_1\le\infty,-\infty\le x_2,x_3\le\infty)$, caused by the action of the internal heat source $F(x,t);\ x\equiv(x_1,x_2,x_3);\quad x\in V$ with the following homogeneous mechanical boundary conditions

$$\sigma_{11}=U_2=U_3=0 \tag{6}$$

together with the boundary heat flux

$$a[\partial T(y,t)/\partial n]=S(y,t),\ y\equiv(0,y_2,y_3),\quad y\in\Gamma \tag{7}$$

at the boundary plane $\Gamma\equiv(y_1=0,\quad -\infty<y_2,y_3<\infty)$.

The initial conditions with respect to the dynamical thermo elastic displacements, their rates and temperature are assumed to be homogeneous

$$U_i(x,t)\big|_{t=0}=f_i(x)=0;\quad \partial U_i(x,t)/\partial t\big|_{t=0}=g_i(x)=0;\quad T(x,t)\big|_{t=0}=\Phi_0(x)=0. \tag{8}$$

To solve this problem, using traditional scheme, it is necessary first to integrate the equation

$$(\partial/\partial t-a\nabla^2)T(x,t)=F(x,t) \tag{9}$$

with the boundary and initial conditions in Eqs (7) (8) with respect to temperature. So, at the first stage from BVP given in Eqs (7)-(9) the temperature field is determined. Then, at the second stage, it is necessary to solve the respective BVP of the dynamical theory of elasticity for the half-space and to determine the field of the thermo elastic displacements $U_i(x,t)$. To do this we need to solve the equations:

$$\mu\nabla^2 U_i(x,t)+(\lambda+\mu)\Theta_{,i}(x,t)-\rho\,\partial^2 U_i(x,t)/\partial t^2-\gamma T_{,i}(x,\xi)=0 \tag{10}$$

with initial and boundary conditions in Eqs (6), (8) and (10). Also, one can use the following integral formula:

$$U_k(\xi,t)=\int_0^t d\tau\int_V F(z,\tau)U_k(z,\xi;t-\tau)dV(z)+a\int_0^t d\tau\int_\Gamma S(y,\tau)U_k(y,\xi;t-\tau)d\Gamma(y), \tag{11}$$

obtained from proposed general integral formula in Eq (5). To determine the influence functions $U_k(z,\xi;t-\tau)$ one should to construct the Green's function $G(z,\xi;t-\tau)$, first. To do this it is necessary to solve the following BVP in heat conduction for the half-space:

$$\begin{gathered}(\partial/\partial t-a\nabla_x^2)G(x,\xi;t-\tau)=\delta(x-\xi)\delta(t-\tau);\,t>\tau;\quad G(x,\xi;t-\tau)\big|_{t-\tau=0}=0;\ ;G(y,\xi;t-\tau)=0\\ x\equiv(x_1,x_2x_3),\,x\in V;\quad y\equiv(0,y_2y_3),\,x\in V;\,y\in\Gamma.\end{gathered} \tag{12}$$

The expression of the Green's function obtained by reflection method is written in the form:

$$\begin{gathered}G(x,\xi;t-\tau)=\left[8\sqrt{\pi^3a^3(t-\tau)^3}\right]^{-1}\left[e^{-\left[R^2(x,\xi)/4\pi(t-\tau)\right]}-e^{-\left[R_1^2(x,\xi)/4\pi(t-\tau)\right]}\right],\\ R=R(x,\xi);\,R_1=R(x,\xi_1^\bullet);\,\xi_1^\bullet=(-\xi_1,\xi_2\xi_3).\end{gathered} \tag{13}$$

Then, in accordance with the formula in Eq (1), at second steep it is necessary to construct the Green's function $\Theta^{(k)}$. To do this one must solve the following differential equations:

$$\begin{gathered}\left(\rho(\lambda+2\mu)^{-1}\partial^2/\partial t^2-a\nabla_x^2\right)\Theta^{(k)}(x,\xi;t-\tau)=(\lambda+2\mu)^{-1}(\partial/\partial x_i)[\delta(x-\xi)\delta(t-\tau)];\,t>\tau;\\ \Theta^{(k)}(x,\xi;t-\tau)\big|_{t-\tau=0}=0;\quad \partial\Theta^{(k)}(x,\xi;t-\tau)/\partial t\big|_{t-\tau=0}=0;\ x\equiv(x_1,x_2x_3),\,x\in V\end{gathered} \tag{14}$$

with mechanical boundary conditions in Eq (6). The representation for $\Theta^{(k)}$ is the following:

$$\begin{gathered}\Theta^{(k)}(x,\xi;t-\tau)=-(\lambda+2\mu)^{-1}(\partial/\partial\xi_k)G_\Theta(x,\xi;t-\tau)+\\ \int_o^t ds\int_\Gamma\left[\frac{\partial\Theta^{(k)}(y,\xi;t-\tau)}{\partial n_y}-\Theta^{(k)}(y,\xi;t-\tau)\frac{\partial}{\partial n_y}\right]G_\Theta(y,\xi;t-\tau)d\Gamma(y),\end{gathered} \tag{15}$$

where function $G_\Theta(x,\xi;t-\tau)$ is the respective Green's function for Eq (14). One can be shown, that from conditions in Eq (6) follows that $\Theta^{(k)}(y,\xi;t-\tau)\big|_\Gamma=0$. To prove this we use the Hook's low and boundary conditions for the stresses

$$\sigma_{11}^{(k)}=2\mu U_{1,1}^{(k)}+\lambda\Theta^{(k)}=(\lambda+2\mu)\Theta^{(k)}-2\mu\left(U_{2,2}^{(k)}+U_{3,3}^{(k)}\right)=0;\quad\Rightarrow\Theta^{(k)}(y,\xi;t-\tau)\big|_\Gamma=0, \tag{16}$$

because from the boundary conditions for displacements $U_2^{(k)}=U_3^{(k)}=0$ follows $U_{2,2}^{(k)}=U_{3,3}^{(k)}=0$ on boundary Γ. So, to construct the function $\Theta^{(k)}$ we must to introduce in Eq (15) the conditions

$$\Theta^{(k)}(y,\xi;t-\tau)\big|_\Gamma=0;\quad G_\Theta(y,\xi;t-\tau)\big|_\Gamma=0 \tag{17}$$

and to construct the Green's function G_Θ. To do this we have to solve the following BVP:

$$\begin{gathered}\left(\rho(\lambda+2\mu)^{-1}\partial^2/\partial t^2-a\nabla_x^2\right)G_\Theta(x,\xi;t-\tau)=\delta(x-\xi)\delta(t-\tau);\,t>\tau;\\ G_\Theta(x,\xi;t-\tau)\big|_{t-\tau=0}=0;\quad\partial G_\Theta(x,\xi;t-\tau)/\partial t\big|_{t-\tau=0}=0;\ G_\Theta(y,\xi;t-\tau)\big|_\Gamma=0;\quad x\equiv(x_1,x_2x_3),\,x\in V\end{gathered} \tag{18}$$

Having constructed the function G_Θ by reflection method (V. Budac, A. Samarskii, A. Tihonov, 1980; A. Kartashov, 1980) and substituting it and results (17) in Eq (15) we obtain the function $\Theta^{(k)}$ in the form:

$$\begin{gathered}\Theta^{(k)}(x,\xi;t-\tau)=-[4\pi(\lambda+2\mu)]^{-1}\frac{\partial}{\partial\xi_k}\left[\delta\left(\frac{R(x,\xi)}{c}-(t-\tau)\right)\frac{1}{R(x,\xi)}-\delta\left(\frac{R_1(x,\xi)}{c}-(t-\tau)\right)\frac{1}{R_1(x,\xi)}\right];\\ c=\sqrt{\rho^{-1}(\lambda+2\mu)}\end{gathered} \tag{19}$$

Finally, if we substitute Eq (19) in general formula (1) and calculate the volume integral, we obtain the dynamical thermo elastic influence functions in the half-space *V* in the form:

$$U_k(x,\xi;t-\tau)=\gamma\int_0^t ds\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dz_2dz_3\int_0^{\infty}G(x,z;t-s)\Theta^{(k)}(z,\xi;s-\tau)dz_1=\gamma\frac{\partial}{\partial\xi_k}(\Phi-\Phi_1). \tag{20}$$

Here the function Φ is determined by the expression

$$\Phi\equiv\Phi(x,\xi;t-\tau)=\frac{m}{4\pi R}\left\{\left(e^{(t^\bullet-\tau^\bullet)-r}-1\right)H\left((t^\bullet-\tau^\bullet)-r\right)-\left[U(r,t^\bullet-\tau^\bullet)-erfc\left(r/2\sqrt{t^\bullet-\tau^\bullet}\right)\right]\right\} \tag{21}$$

$$\begin{gathered}U(r,t-\tau)=\frac{e^{t^\bullet-\tau^\bullet}}{2}\left[e^{r}erfc\left(r/2\sqrt{t^\bullet-\tau^\bullet}+\sqrt{t^\bullet-\tau^\bullet}\right)+e^{-r}erfc\left(r/2\sqrt{t^\bullet-\tau^\bullet}-\sqrt{t^\bullet-\tau^\bullet}\right)\right]\\ r=\left(a^{-1}c\right)R(x,\xi);\qquad t^\bullet-\tau^\bullet=\left(a^{-1}c^2\right)(t-\tau);\quad m=\gamma c^{-2}\rho^{-1}\end{gathered} \tag{22}$$

where the Heaviside function H is given by the formula

$$H\equiv H\left((t^\bullet-\tau^\bullet)-r\right)=\begin{Bmatrix}0;\ (t^\bullet-\tau^\bullet)\le r\quad or\quad (t-\tau)\le R(x,\xi)/c\\ 1;\ (t^\bullet-\tau^\bullet)> r\quad or\quad (t-\tau)> R(x,\xi)/c\end{Bmatrix} \tag{23}$$

The function Φ_1 in Eq (20) can be obtained from the function Φ in Eqs (21) and (22) by replacing the point $\xi\equiv(\xi_1,\xi_2,\xi_3)$ with the symmetric point $\xi_1^\bullet\equiv(-\xi_1,\xi_2,\xi_3)$. If, based on Eq (20) we calculate the limit $\lim_{x\to y}U_k(x,\xi;t-\tau)$ and substitute this value in Eq (11), then we obtain the following integral formula for thermo elastic dynamical displacements in half-space *V*:

$$\begin{gathered}U_k(\xi,\tau)=\gamma\int_0^t d\tau\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dz_2dz_3\int_0^{\infty}F(z,\tau)\frac{\partial}{\partial\xi_k}\left[\Phi(z,\xi;s-\tau)-\Phi_1(z,\xi;s-\tau)\right]dz_1+\\ 2a\gamma\int_0^t d\tau\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}S(y,\tau)\frac{\partial^2}{\partial\xi_k\partial y_1}\Phi(y,\xi;s-\tau)dy_2dy_3.\end{gathered} \tag{24}$$

The formula in Eq (24) presents an analog of the Poisson integral formula for a half-space in uncoupled dynamical thermo elasticity.

CONCLUSIONS

The advantage of proposed method is that it allows us to unite the two-stage process of solving the BVP in the theory of dynamical thermal stresses into the single ones. Closed form concrete solutions for functions $U_k(x,\xi;t-\tau)$ using formulas in Eq (1), as the formulas in Eqs (20)-(23) for half-space, as well as for solutions in closed form of Poisson's type integrals in Eq (24) are very height appreciate by the specialists in the area of applied mathematical physics, applied mechanics and dynamical thermo elasticity as a considerable contribution to these areas of research. Such kind of solutions plays a very important role in development and checking of numerical and analytical methods to solve these BVP.

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