# Perron-Frobenius dynamics for Markov chains

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This talk is dedicated to studying the problem of asymptotic behavior of trajectories of linear dynamical system

$$x(t+1) = Ax(t) \tag{1}$$

with discrete time $t \in \mathbb{Z}_+$ and non-negative stochastic matrix $A = (a_{ij})_{i,j=1}^n$, i.e., with the condition

$$a_{ij} \geq 0 \ \text{ and } \ \sum_{i=1}^{n} a_{ij} = 1 (\forall\, i, j = 1, 2, \ldots, n) \tag{2}$$

on the set $M := \{x \in \mathbb{R}^n |\ x_i \geq 0,\ \sum_{i=1}^n = 1\}$. We also consider the generalization of this problem for non-stationary (non-autonomous) linear systems

$$x(t+1) = A(t)x(t), \tag{3}$$

for some classes of non-linear systems

$$\Delta x(t) = f(t, x(t)), \tag{4}$$

where $\Delta x(t) := x(t+1) - x(t))$, and for abstract discrete non-autonomous dynamical systems.

**Perron-Frobenius dynamics.**

Let $A = (a_{ij})_{i,j=1}^n$ be a stochastic matrix. The matrix $A$ can be considered as the transition matrix for a Markov process acting on a set of $n$ states $\{1, 2, \ldots, n\}$.

Let $M := \{x \in \mathbb{R}^n_+ |\ \sum_{i=1}^n x_i = 1\}$. Let $A = (a_{ij})_{i,j=1}^n$ be a stochastic matrix. Since

$$\sum_{i=1}^{n} (Ax)_i = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_j = \sum_{j=1}^{n} \Big( \sum_{i=1}^{n} a_{ij} \Big) x_j = 1,$$

then $Ax \in M$ for any $x \in M$. The positive iterations of the mapping $x \to Ax$ ($x \in M$) defines a discrete semi-cascade on $M$. Note that the set $M$ is a compact and convex subset of $\mathbb{R}^n_+$.

Denote by $Fix(A)$ the set of all fixed points of semi-cascade $(M, A)$.

**Theorem 1.** *Let $A = (a_{ij})_{i,j=1}^n \in [\mathbb{R}^n]$ be a stochastic nonnegative matrix with $a_{ii} > 0$ for any $i = 1, , \ldots, n$. Then the following statements hold:*

*(1) the semi-cascade $(M, A)$ has a nonempty and compact set of fixed points $Fix(A) \subseteq M$;*

*(2) for every $x \in M$ there exists $\lim_{k\to\infty} A^k x = p_x$ and $p_x \in Fix(A)$ for any $x \in M$;*

*(3) every fixed point $p \in Fix(A)$ of the cascade $(M, A)$ is positively stable, ie.e, for any positive number $\varepsilon$ ther exist a positive number $\delta = \delta(\varepsilon)$ such that $|A^k x - p| < \varepsilon$ for any $k \in \mathbb{Z}_+$, whenever $|x - p| < \delta$ ($x \in M$);*

*(4) the semi-cascade $(M, A)$ is compact dissipative and its Levinson center $J$ coincide with the set $Fix(A)$;*

*(5) $Fix(A) = \bigcap_{k=0}^{\infty} A^k M$ and it is convex;*

Denote by $Int(M)$ the interior of the set $M$.

**Theorem 2.** *Suppose that the stochastic matrix $A$ is positive ($a_{ij} > 0$ for any $i, j = 1, \ldots, n$), then the following statements hold:*

*(1) the semi-cascade $(M, A)$ has a unique point $p \in M$;*

*(2) the vector $p \in M$ is positive, i.e., $p_i > o$ for any $i = 1, \ldots, n$;*

*(3) $p$ is globally asymptotically stable, i.e.,*

*(a) for any positive number $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that $|x - p| < \delta$ ($x \in M$) implies $|A^k x - p| < \varepsilon$ for any $k \in \mathbb{Z}_+$; and*

*(b)*

$$\lim_{k\to\infty} A^k x = p$$

*for any $x \in M$.*

**Remark 3.** Notice that for non-negative stochastic matrix $A = (a_{ij})_{i,j=1}^n$ with $a_{ii} > 0$ the set $Fix(A)$, generally speaking, it is not reduced to a single point.

This statement can be confirmed by following example

**Example 4.** Consider the following stochastic matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1/2 & 1 \end{pmatrix}.$$

It easy to check that

$$\lim_{k\to\infty} A^k \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ 1 - x_1 \end{pmatrix}$$

for any $x_1 \in [0,1]$, because $x_1 + x_2 + x_3 = 1$. Thus we have $Fix(A) := \{p_\alpha : \ \alpha \in [0,1]\}$, where

$$p_\alpha := \begin{pmatrix} \alpha \\ 0 \\ 1 - \alpha \end{pmatrix},$$

i.e., $Fix(A)$ coincides with an entire (nontrivial) segment in $M$.

**References:**

1. David N. Cheban, *Nonautonomous Dynamics:* Nonlinear oscillations and Global attractors. Springer Nature Switzerland AG 2020, xxii+ 434 pp.

2. B. M. Levitan and V. V. Zhikov, *Almost Periodic Functions and Differential Equations.* Moscow State University Press, Moscow, 1978, 204 pp. (in Russian). [English translation: Almost Periodic Functions and Differential Equations. Cambridge Univ. Press, Cambridge, 1982, xi+217 p.]

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