A Note on some Open Problems in Topological Algebra

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Objects of topological algebra, defined as a certain combination of algebraic and topological structures, often give rise to original and unusual questions. A special additional topological property of many topological spaces of this kind is homogeneity. A topological space X is called *homogeneous* if, for any x, y in X there exists a homeomorphism f of X onto itself such that $f(x) = y$ and $f(X) = X$. Clearly all topological groups, in particular, all linear topological spaces are homogeneous. This simple fact provides us with a natural way to construct many homogeneous compact spaces, since there are many compact topological groups. Some of them are non-metrizable. This occurs precisely when a compact topological group is not sequential - that is, when its topology cannot be described in terms of converging sequences. In this connection, it is especially interesting that every infinite compact topological group has many non-trivial converging sequences. But the following question, posed by Walter Rudin more than 60 years ago, is still open:

Problem 1. *(W. Rudin) Is it true that every infinite homogeneous compact Hausdorrf space contains a non-trivial converging sequence?*

Many compact topological groups contain, in fact, dense sequential subgroups. In this connection I have formulated, about forty years ago, the next question, which seems still to be not answered:

Problem 2. *(A.V. Arhangel'skii) Is it true that every infinite compact topological group contains a dense sequential subspace?*

It follows from the results obtained by me in early seventies rhat the next statement holds:

Theorem 3. *Under CH, every homogeneous sequential compact Hausdorff space is first countable, and hence, its cardinality does not exceed* 2^{ω}.

In this connection, the following questions arise:

Problem 4. *(A.V. Arhangel'skii) Is it true in ZFC that every homogeneous sequential compact Hausdorff space is first countable?*

I also want to mention another open question, formulated by me in eighties:

Problem 5. *Suppose that X is a paracompact p-space. Then is its free topological group $F(X)$ (or the Abelian version of it) paracompact?*

It was shown by me in eighties that if X is metrizable, then the answer is "yes".

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