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Max-Erlang and Min-Erlang power series distributions as two new families of lifetime distribution

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Abstract. The distribution of the minimum and maximum of a random number of independent, identically Erlang distributed random variables are studied. Some particular cases of such kind of lifetime distributions are discussed too.

Mathematics subject classification: 62K10, 60N05.
Keywords and phrases: Power series distribution, distribution of the maximum, distribution of the minimum, Erlang distribution.

1 Preliminary results

In this paper we present two new families of distribution, namely the Max-Erlang power series (MaxErlPS) distribution, respectively the Min-Erlang power series (MinErlPS) distribution. These are obtained by mixing the distribution of the maximum or the minimum of a fixed number of independent Erlang distributed random variables, where the combination is obtained by the techniques that have been treated by Adamidis and Loukas (1998, [1]) and more generally by Chahkandi and Ganjali (2009, [6]) or Baretto-Souza and Cribari (2009, [3]). Recently, the new distributions that model the reliability systems were obtained by exponential distribution with several discrete distributions (the families of the power series distributions). For example, the distribution of the minimum of a sample of random size with the exponential distribution was obtained. In this connection, the geometric distribution, the Poisson distribution and the logarithmic distribution were considered by Adamidis and Loukas (1998, [1]), Kus (2007, [10]), Tahmasbi and Rezaei (2008, [19]).

Then, the previous results have been generalized by Chahkandi and Ganjali (2009, [6]) using the compounding exponential distribution with the power series distribution, thus obtaining the exponential power series distribution (EPS) type. Later, Morais and Baretto-Souza (2011, [16]) replaced the exponential distribution with the Weibull power series distribution (WPS) of the minimum of a sequence of the independent and identically distributed random variables (i. i. d. r. v.) in a random number, studying the distribution of the strength of 1.5 cm glass fibers. The case of the maximum has been discussed and analyzed by Munteanu (2013, [17]), introducing the Max Weibull power series (MaxWPS) distribution that particularizes the complementary exponential geometric (CEG) distribution introduced by

Louzada, Roman and Cancho (2011, [14]), the complementary exponential Poisson (CEP) distribution introduced by Cancho, Louzada and Barriga (2011, [5]) and the complementary exponential logaritmic (CEL) distribution introduced by Flores, Borges, Cancho and Louzada (2013, [8]).

The geometric distribution which contains a power parameter was considered by Adamidis, Dimitrakopoulou and Loukas (2005, [2]) and later generalized by Silva, Baretto-Souza and Cordeiro (2010, [18]). The Poisson distribution being treated by Cancho, Louzada and Barriga (2011, [5]) and which then was generalized by Cordeiro, Rodriques and Castro (2011, [7]) as the COMPoisson distribution, this contains a parameter power, the power series distribution type is not considered because the maximum number will being one deterministic.

The methodology and techniques used in this article are shown in the paper of Leahu, Munteanu and Cataranciuc (2013, [12]), general framework illustrating the particular cases treated in the works of Adamidis and Loukas (1998, [1]), Kus (2007, [10]), Tahmasbi and Rezaei (2008, [19]), Leahu and Lupu (2010, [11]), Baretto-Souza, Morais and Cordeiro (2011, [4]), Morais and Baretto-Souza (2011, [16]), Cancho, Louzada and Barriga (2011, [5]), Louzada, Roman and Cancho (2011, [14]), Flores, Borges, Cancho and Louzada (2013, [8]).

Let ′ s consider r.v. Z such that $\mathbb{P}(Z \in \{1,2,\ldots\}) = 1$.

Definition 1 (see [9]). We say that r. v. Z has a power series distribution if:

$$\mathbb{P}(Z = z) = \frac{a_z \Theta^z}{A(\Theta)},\ z = 1,2,\ldots;\ \Theta \in (0,\tau), \tag{1}$$

where $a_1, a_2, \ldots$ are nonnegative real numbers, τ is a positive number bounded by the convergence radius of power series (*series function*) $A(\Theta) = \sum_{z \geq 1} a_z \Theta^z$, $\forall\ \Theta \in (0,\tau)$, and Θ is *power parameter* of the distribution (Table 1).

PSD denotes the power series distribution functions families. If the r. v. Z has the distribution from relationship (1), then we write that $Z \in PSD$.

Table 1. The representative elements of the PSD families for various truncated distributions

Distribution	a_z	Θ	$A(\Theta)$	τ
$Binom^*(n,p)$	$\binom{n}{z}$	$\frac{p}{1-p}$	$(1+\Theta)^n - 1$	∞
$Poisson^*(\alpha)$	$\frac{1}{z!}$	α	$e^{\Theta} - 1$	∞
$Log(p)$	$\frac{1}{z}$	p	$-ln(1-\Theta)$	1
$Geom^*(p)$	1	$1-p$	$\frac{\Theta}{1-\Theta}$	1
$Pascal(k,p)$	$\binom{z-1}{k-1}$	$1-p$	$\left(\frac{\Theta}{1-\Theta}\right)^k$	1
$Bineg^*(k,p)$	$\binom{z+k-1}{z}$	p	$(1-\Theta)^{-k} - 1$	1

2 On the properties of the Max-Erlang and the Min-Erlang power series distributions

We consider that $X_i \sim Erlang(k, \lambda)$, $k \in \mathbb{N}$, $k \geq 1$, $\lambda > 0$, where $(X_i)_{i\geq 1}$ i. i. d. r. v. with the distribution function $F_{X_i}(x) \equiv F_{Erl}(x) = 1 - \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!} e^{-\lambda x}$, $x > 0$ and the pdf $f_{X_i}(x) \equiv f_{Erl}(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!}$, $x > 0$. We note that $U_{Erl} = \max\{X_1, X_2, \ldots, X_Z\}$ and $V_{Erl} = \min\{X_1, X_2, \ldots, X_Z\}$.

The results in this section are obtained using the general framework of the work [12], for which reason some proofs are not presented.

Proposition 1. *If r. v. $U_{Erl} = \max\{X_1, X_2, \ldots, X_Z\}$ and $V_{Erl} = \min\{X_1, X_2, \ldots, X_Z\}$, where $(X_i)_{i\geq 1}$ are nonnegative i. i. d. r. v., $X_i \sim Erlang(k, \lambda)$, $k \in \mathbb{N}$, $k \geq 1$, $\lambda > 0$ and $Z \in PSD$ with $\mathbb{P}(Z = z) = \frac{a_z \Theta^z}{A(\Theta)}$, $z = 1, 2, \ldots$; $\Theta \in (0, \tau)$, $\tau > 0$, r. v. $(X_i)_{i\geq 1}$ and Z being independent, then the distribution functions of the r. v. U_{Erl}, respectively V_{Erl} are the following:*

$$U_{Erl}(x) = \frac{A\left[\Theta\left(1 - e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)\right]}{A(\Theta)}, \ x > 0, \tag{2}$$

$$V_{Erl}(x) = 1 - \frac{A\left[\Theta e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right]}{A(\Theta)}, \ x > 0. \tag{3}$$

We denote a r. v. U_{Erl} following Max-Erlang power series (MaxErlPS) distribution with parameters k, λ and Θ by $U_{Erl} \sim \text{MaxErlPS}(k, \lambda, \Theta)$, respectively a r. v. V_{Erl} following Min-Erlang power series (MinErlPS) distribution with parameters k, λ and Θ by $V_{Erl} \sim \text{MinErlPS}(k, \lambda, \Theta)$.

The following results characterize the survival functions and the probability density functions (pdf) for the maximum, respectively minimum of a sequence of independent Erlang distributed random variables in a random number.

Consequence 1. *If r. v. $U_{Erl} \sim \text{MaxErlPS}(k, \lambda, \Theta)$ and $V_{Erl} \sim \text{MinErlPS}(k, \lambda, \Theta)$, then the survival functions of the r. v. U_{Erl}, respectively V_{Erl} are the following:*

$$S_{U_{Erl}}(x) = 1 - \frac{A\left[\Theta\left(1 - e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)\right]}{A(\Theta)}, \ x > 0, \tag{4}$$

$$S_{V_{Erl}}(x) = \frac{A\left[\Theta e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right]}{A(\Theta)}, \ x > 0. \tag{5}$$

Consequence 2. *The pdf′ s of the r. v. U_{Erl}, respectively V_{Erl} are the following:*

$$u_{Erl}(x) = \frac{\Theta\lambda^k x^{k-1} e^{-\lambda x} A' \left[\Theta\left(1 - e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)\right]}{A(\Theta)},\ x > 0 \tag{6}$$

and

$$v_{Erl}(x) = \frac{\Theta\lambda^k x^{k-1} e^{-\lambda x} A' \left[\Theta e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right]}{A(\Theta)},\ x > 0. \tag{7}$$

Proposition 2. *The hazard rates for the r. v. U_{Erl}, respectively V_{Erl} are characterized by the following relations:*

$$h_{U_{Erl}}(x) = \frac{u_{Erl}(x)}{1 - U_{Erl}(x)} = \frac{\Theta\lambda^k x^{k-1} e^{-\lambda x} A' \left[\Theta\left(1 - e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)\right]}{A(\Theta) - A\left[\Theta\left(1 - e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)\right]}$$

and

$$h_{V_{Erl}}(x) = \frac{v_{Erl}(x)}{1 - V_{Erl}(x)} = \frac{\Theta\lambda^k x^{k-1} e^{-\lambda x} A' \left[\Theta e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right]}{A\left[\Theta e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right]}.$$

The next result shows a characteristic of the MaxErlPS and MinErlPS distributions.

Proposition 3. *If $(X_i)_{i\geq 1}$ is a sequence of independent, identically Erlang distributed r. v., with parameters $\lambda > 0$, $k \in \{1, 2, \ldots\}$ and $Z \in PSD$ with $\mathbb{P}(Z = z) = \frac{a_z\Theta^z}{A(\Theta)}$, where $(a_z)_{z\geq 1}$ is a sequence of nonnegative real numbers, $A(\Theta) = \sum_{z\geq 1} a_z\Theta^z$, $\forall\ \Theta \in (0, \tau)$, then*

$$\lim_{\Theta\to 0^+} U_{Erl}(x) = \left[1 - \sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}e^{-\lambda x}\right]^m,\ x > 0,$$

considering $m = \min\{n \in \mathbb{N}^*,\ a_n > 0\}$.

Proof. By applying the l′ Hospital rule m-time, we have:

$$\begin{aligned}\lim_{\Theta\to 0^+} U_{Erl}(x) &= \lim_{\Theta\to 0^+}\frac{A^{(m)}\left[\Theta\left(1 - e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)\right]\cdot\left(1 - e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)^m}{A^{(m)}(\Theta)} \\ &= \frac{m!a_m\left(1 - e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)^m}{m!a_m} = \left(1 - e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)^m,\ x > 0\end{aligned}$$

and $m = \min\{n \in \mathbb{N}^*,\ a_n > 0\}$. □

Applying the same method of proof of Proposition 3, we obtain:

Proposition 4. *Under the conditions of the Proposition 3, when $\Theta \to 0^+$, we have that*

$$\lim_{\Theta \to 0^+} V_{Erl}(x) = 1 - \left[\sum_{i=0}^{k-1} \frac{(\lambda x)^i}{i!} e^{-\lambda x} \right]^l, \ x > 0,$$

where $l = \min\{n \in \mathbb{N}^*, \ a_n > 0\}$.

Consequence 3. *The r^{th} moments, $r \in \mathbb{N}$, $r \geq 1$ of the r.v. $U_{Erl} \sim \mathrm{MaxErlPS}(k, \lambda, \Theta)$ and $V_{Erl} \sim \mathrm{MinErlPS}(k, \lambda, \Theta)$ are given by*

$$\mathbb{E} U_{Erl}^r = \sum_{z \geq 1} \frac{a_z \Theta^z}{A(\Theta)} \mathbb{E} \left[\max\{X_1, \ X_2, \ldots, X_z\} \right]^r \tag{8}$$

and

$$\mathbb{E} V_{Erl}^r = \sum_{z \geq 1} \frac{a_z \Theta^z}{A(\Theta)} \mathbb{E} \left[\min\{X_1, \ X_2, \ldots, X_z\} \right]^r, \tag{9}$$

where pdf' s $f_{\max\{X_1, \ X_2, \ldots, X_z\}}(x) = z f_{Erl}(x) \left[F_{Erl}(x)\right]^{z-1}$ and $f_{\min\{X_1, \ X_2, \ldots, X_z\}}(x) = z f_{Erl}(x) \left[1 - F_{Erl}(x)\right]^{z-1}$.

The distribution functions and pdf' s of the r.v. $U_{Erl} \sim \mathrm{MaxErlPS}(k, \lambda, \Theta)$ for different combinations of the r.v. $Z \in PSD$ (Table 1), are the following:

- $Z \sim Binom^*(n, p)$; $U_{ErlB} \sim \mathrm{MaxErlB}(k, \lambda, n, p)$, $\lambda > 0$; $k, n \in \{1, 2 \ldots\}$; $p \in (0, 1)$:

$$U_{ErlB}(x) = \frac{\left(1 - p e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right)^n - 1}{1 - (1-p)^n}, \ x > 0,$$

$$u_{ErlB}(x) = \frac{n p \lambda^k x^{k-1} e^{-\lambda x} \left(1 - p e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right)^{n-1}}{1 - (1-p)^n}, \ x > 0.$$

- $Z \sim Poisson^*(\alpha)$; $U_{ErlP} \sim \mathrm{MaxErlP}(k, \lambda, \alpha)$, $\lambda, \alpha > 0$; $k \in \{1, 2 \ldots\}$:

$$U_{ErlP}(x) = \frac{e^{-\alpha e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}} - e^{-\alpha}}{1 - e^{-\alpha}}, \ x > 0,$$

$$u_{ErlP}(x) = \frac{\alpha \lambda^k x^{k-1} e^{-\lambda x - \alpha e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}}}{1 - e^{-\alpha}}, \ x > 0.$$

- $Z \sim Log(p)$; $U_{ErlLog} \sim \text{MaxErlLog}(k, \lambda, p),\ \lambda > 0;\ k \in \{1, 2 \ldots\};\ p \in (0, 1)$:

$$U_{ErlLog}(x) = \ln\left[1 - p + p\, e^{-\lambda x} \sum_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right]^{-a},\ x > 0,$$

$$u_{ErlLog}(x) = \frac{ap\lambda^k x^{k-1} e^{-\lambda x}}{\left[1 - p + p\, e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right]},\ x > 0,$$

where $a = -1/\ln(1-p)$.

- $Z \sim Geom^*(p)$; $U_{ErlG} \sim \text{MaxErlG}(k, \lambda, p),\ \lambda > 0;\ k \in \{1, 2 \ldots\};\ p \in (0, 1)$:

$$U_{ErlG}(x) = \frac{p\left(1 - e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right)}{p + (1-p)e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}},\ x > 0,$$

$$u_{ErlG}(x) = \frac{p\lambda^k x^{k-1} e^{-\lambda x}}{\left[p + (1-p)e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right]^2},\ x > 0.$$

- $Z \sim Pascal(k^\star, p)$; $U_{ErlPas} \sim \text{MaxErlPas}(k, \lambda, k^\star, p),\ \lambda > 0;\ k,\ k^\star \in \{1, 2 \ldots\};$ $p \in (0, 1)$:

$$U_{ErlPas}(x) = \left[\frac{p\left(1 - e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right)}{p + (1-p)e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}}\right]^{k^\star},\ x > 0,$$

$$u_{ErlPas}(x) = \frac{k^\star p^{k^\star} \lambda^k x^{k-1} e^{-\lambda x} \left(1 - e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right)^{k^\star - 1}}{\left[p + (1-p)e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right]^{k^\star + 1}},\ x > 0.$$

- $Z \sim Bineg^*(k^\star, p)$; $U_{ErlBineg} \sim \text{MaxErlBineg}(k, \lambda, k^\star, p),\ \lambda > 0;\ k,\ k^\star \in \{1, 2 \ldots\};\ p \in (0, 1)$:

$$U_{ErlBineg}(x) = \frac{\left(1 - p + pe^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right)^{-k^\star} - 1}{(1-p)^{-k^\star} - 1},\ x > 0,$$

$$u_{ErlBineg}(x) = \frac{k^\star p \lambda^k x^{k-1} e^{-\lambda x} \left(1 - p + pe^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right)^{-k^\star - 1}}{(1-p)^{-k^\star} - 1},\ x > 0.$$

The above results shows that the following result is valid:

Proposition 5. *If* $(X_i)_{i\geq 1}$, $X_i \sim Erlang(k,\lambda)$, $k \in \{1,2,\ldots\}$, $\lambda > 0$ *and* $(Y_j)_{j\geq 1}$ *are i. i. d. r. v.,* $Y_j \sim MaxErlG(k,\lambda,p)$, $p\in(0,1)$, *then the r. v.* $\max\{Y_1,Y_2,\ldots,Y_{k^\star}\}$ *has the same distribution as the r.v.* $\max\{X_1,X_2,\ldots,X_Z\}$, *where* $Z\sim Pascal(k^\star,p)$, $k^\star\in\{1,2,\ldots\}$, $p\in(0,1)$.

Proof. Indeed, it is known that if $(Y_j)_{j\geq 1}$ are independent r.v. Max-Erlang-Geometric (MaxErlG) distributed, with the distribution function $U_{ErlG}(x)$, $x>0$, we have:

$$F_{\max\{Y_1,Y_2,\ldots,Y_{k^\star}\}}(x) = \left(U_{ErlG}(x)\right)^{k^\star} = U_{ErlPas}(x),\ \forall\, x>0,$$

where $U_{ErlPas}(x)$ represents the distribution function of the Max-Erlang-Pascal (MaxErlPas) distribution. □

The distribution functions and pdf ′ s of the r. v. $V_{Erl}\sim$ MinErlPS(k,λ,Θ) for different combinations of the r. v. $Z\in PSD$ (Table 1), are the following:

- $Z\sim Binom^*(n,p)$; $V_{ErlB}\sim$ MinErlB(k,λ,n,p) $\lambda>0$; $k,n\in\{1,2\ldots\}$; $p\in(0,1)$:

$$V_{ErlB}(x)=\frac{1-\left(1-p+pe^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)^n}{1-(1-p)^n},\ x>0,$$

$$v_{ErlB}(x)=\frac{np\lambda^k x^{k-1}e^{-\lambda x}\left(1-p+pe^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)^{n-1}}{1-(1-p)^n},\ x>0.$$

- $Z\sim Poisson^*(\alpha)$; $V_{ErlP}\sim$ MinErlP(k,λ,α), $\lambda,\alpha>0$; $k\in\{1,2\ldots\}$:

$$V_{ErlP}(x)=\frac{1-e^{-\alpha\left(1-e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)}}{1-e^{-\alpha}},\ x>0,$$

$$v_{ErlP}(x)=\frac{\alpha\lambda^k x^{k-1}e^{-\lambda x-\alpha+\alpha e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}}}{1-e^{-\alpha}},\ x>0.$$

- $Z\sim Log(p)$; $V_{ErlLog}\sim$ MinErlLog(k,λ,p), $\lambda>0$; $k\in\{1,2\ldots\}$; $p\in(0,1)$:

$$V_{ErlLog}(x)=1+\ln\left[1-p\,e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right]^a,\ x>0,$$

$$v_{ErlLog}(x)=\frac{ap\lambda^k x^{k-1}e^{-\lambda x}}{\left[1-p\,e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right]},\ x>0,$$

  where $a=-1/\ln(1-p)$.

- $Z \sim Geom^*(p)$; $V_{ErlG} \sim \text{MinErlG}(k, \lambda, p),\ \lambda > 0;\ k \in \{1, 2 \ldots\};\ p \in (0, 1):$

$$V_{ErlG}(x) = \frac{1 - e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}}{1 - (1-p)e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}},\ x > 0,$$

$$v_{ErlG}(x) = \frac{p\lambda^k x^{k-1} e^{-\lambda x}}{\left[1 - (1-p)e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right]^2},\ x > 0.$$

- $Z \sim Pascal(k^\star, p)$; $V_{ErlPas} \sim \text{MinErlPas}(k, \lambda, k^\star, p)\ \lambda > 0;\ k,\ k^\star \in \{1, 2 \ldots\};$ $p \in (0, 1):$

$$V_{ErlPas}(x) = \left[\frac{1 - e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}}{1 - (1-p)e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}}\right]^{k^\star},\ x > 0,$$

$$v_{ErlPas}(x) = \frac{k^\star p^{k^\star} \lambda^k x^{k-1} e^{-k^\star \lambda x} \left(\sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right)^{k^\star - 1}}{\left[1 - (1-p)e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right]^{k^\star + 1}},\ x > 0.$$

- $Z \sim Bineg^*(k^\star, p)$; $V_{ErlBineg} \sim \text{MinErlBineg}(k, \lambda, k^\star, p),\ \lambda > 0;\ k,\ k^\star \in \{1, 2 \ldots\};\ p \in (0, 1):$

$$V_{ErlBineg}(x) = \frac{(1-p)^{-k^\star} - \left(1 - pe^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right)^{-k^\star}}{(1-p)^{-k^\star} - 1},\ x > 0,$$

$$v_{ErlBineg}(x) = \frac{k^\star p \lambda^k x^{k-1} e^{-\lambda x} \left(1 - pe^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right)^{-k^\star - 1}}{(1-p)^{-k^\star} - 1},\ x > 0.$$

Figure 1 shows several representations of pdf ′ s of some particular MaxErlPS distributions ($\text{MaxErlB}(k, \lambda, n, p)$, $\text{MaxErlP}(k, \lambda, \alpha)$), for different values of their parameters: $k = 2, \lambda = 3.5, \alpha = 7, n = 21, p = 1/4$.

Figure 2 shows the behavior of the pdf ′ s of the $\text{MinErlB}(k, \lambda, n, p)$, $\text{MinErlP}(k, \lambda, \alpha)$ for some values of the parameters: $k = 2, \lambda = 0.5, \alpha = 5, n = 25, p = 1/6$.

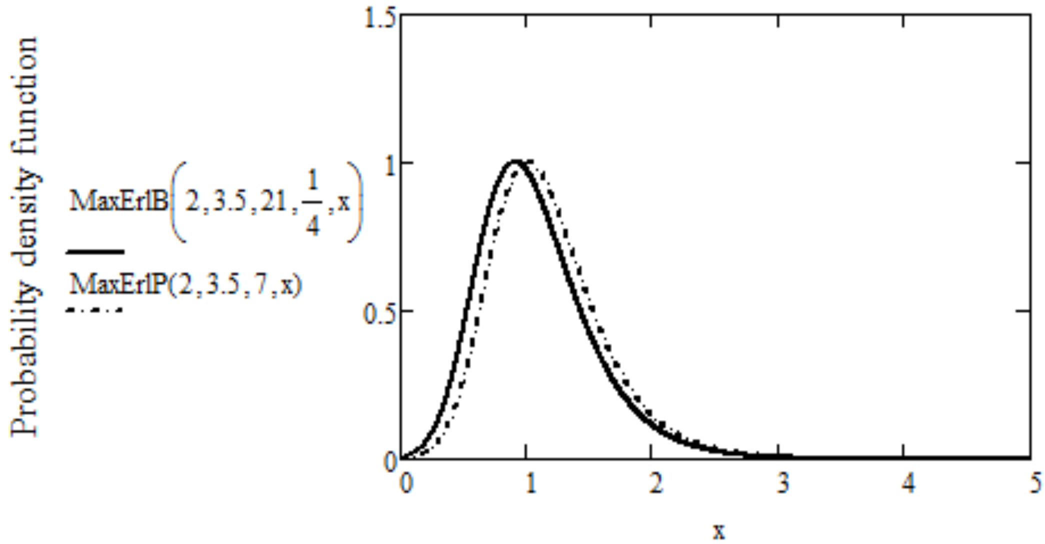

Figure 1. Pdf ′ s for Max-Erlang-Binomial and Max-Erlang-Poisson distributions

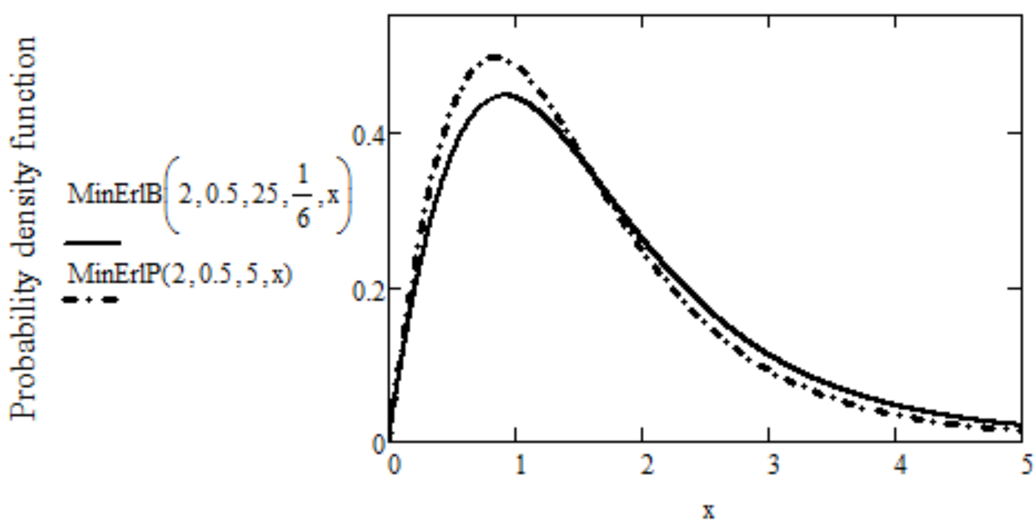

Figure 2. Pdf ′ s for Min-Erlang-Binomial and Min-Erlang-Poisson distributions

3 Special cases

In this section, we shall illustrate the characteristics of four distributions: the Max-Erlang-Binomial (MaxErB) distribution, the Min-Erlang-Binomial (MinErlB) distribution, the Max-Erlang-Poisson (MaxErlP) distribution, respectively the Min-Erlang-Poisson (MinErlP) distribution, so that later we can formulate a Poisson limit theorem.

3.1 The MaxErlB and MinErlB distributions

The MaxErlB and MinErlB distributions is defined by the distribution functions (2) and (3), with $A(\Theta) = (\Theta + 1)^n - 1$, namely:

$$U_{ErlB}(x) = \frac{\left(1 + \Theta - \Theta e^{-\lambda x} \sum_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right)^n - 1}{(1 + \Theta)^n - 1}, \ x > 0 \tag{10}$$

and

$$V_{ErlB}(x) = \frac{(1 + \Theta)^n - \left(1 + \Theta e^{-\lambda x} \sum_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right)^n}{(1 + \Theta)^n - 1}, \ x > 0, \tag{11}$$

where n is integer pozitive.

The survival functions, defined by the relationships (4), (5), for the r.v. U_{ErlB}, respectively V_{ErlB} are the following:

$$S_{U_{ErlB}}(x) = \frac{(1 + \Theta)^n - \left(1 + \Theta - \Theta e^{-\lambda x} \sum_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right)^n}{(1 + \Theta)^n - 1}, \ x > 0$$

and

$$S_{V_{ErlB}}(x) = \frac{\left(1 + \Theta e^{-\lambda x} \sum_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right)^n - 1}{(1 + \Theta)^n - 1}, \ x > 0.$$

By using the relationships (6), (7) and Proposition 2, the pdf ′ s and hazard rates are given by:

$$u_{ErlB}(x) = \frac{n\Theta\lambda^k x^{k-1} e^{-\lambda x} \left(1 + \Theta - \Theta e^{-\lambda x} \sum_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right)^{n-1}}{(1 + \Theta)^n - 1}, \ x > 0,$$

$$v_{ErlB}(x) = \frac{n\Theta\lambda^k x^{k-1} e^{-\lambda x} \left(1 + \Theta e^{-\lambda x} \sum_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right)^{n-1}}{(1 + \Theta)^n - 1}, \ x > 0$$

and

$$h_{U_{ErlB}}(x) = \frac{n\Theta\lambda^k x^{k-1} e^{-\lambda x}\left(1+\Theta-\Theta e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)^{n-1}}{(1+\Theta)^n - \left(1+\Theta-\Theta e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)^n},\ x>0,$$

respectively

$$h_{V_{ErlB}}(x) = \frac{n\Theta\lambda^k x^{k-1} e^{-\lambda x}\left(1+\Theta e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)^{n-1}}{\left(1+\Theta e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)^n},\ x>0.$$

3.2 The MaxErlP and MinErlP distributions

The MaxErlP and MinErlP distributions is defined by the distribution functions (2) and (3) with $A(\Theta)=e^{\Theta}-1$, and $\Theta=\alpha>0$, namely:

$$U_{ErlP}(x) = \frac{e^{-\Theta e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}} - e^{-\Theta}}{1-e^{-\Theta}},\ x>0 \tag{12}$$

and

$$V_{ErlP}(x) = \frac{1-e^{-\Theta\left(1-e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)}}{1-e^{-\Theta}},\ x>0. \tag{13}$$

By using Consequences 1, the survival functions for the r. v. U_{ErlP}, respectively V_{ErlP} are the following:

$$S_{U_{ErlP}}(x) = \frac{1-e^{-\Theta e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}}}{1-e^{-\Theta}},\ x>0$$

and

$$S_{V_{ErlP}}(x) = \frac{e^{-\Theta\left(1-e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)} - e^{-\Theta}}{1-e^{-\Theta}},\ x>0.$$

With the formulas (6), (7) and Proposition 2, the pdf ′ s and the hazard rates are given by:

$$u_{ErlP}(x) = \frac{\Theta\lambda^k x^{k-1} e^{-\lambda x-\Theta e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}}}{1-e^{-\Theta}},\ x>0,$$

$$v_{ErlP}(x) = \frac{\Theta \lambda^k x^{k-1} e^{-\lambda x - \Theta + \Theta e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}}}{1 - e^{-\Theta}}, \ x > 0$$

and

$$h_{U_{ErlP}}(x) = \frac{\Theta \lambda^k x^{k-1} e^{-\lambda x - \Theta e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}}}{1 - e^{-\Theta e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}}}, \ x > 0,$$

respectively

$$h_{V_{ErlP}}(x) = \frac{\Theta \lambda^k x^{k-1} e^{\Theta e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}}}{e^{\Theta e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}}}, \ x > 0.$$

3.3 On the Poisson limit theorem

The next proposition shows that the MaxErlP and MinErlP distributions approximate the MaxErlB, respectively MinErlB distributions under certain conditions.

Proposition 6. ***(Poisson limit theorem)***. *The* MaxErlP *and* MinErlP *distributions can be obtained as limiting of the* MaxErlB*, respectively* MinErlB *distributions with ditribution functions given by (10) and (11) if* $n\Theta \to \alpha > 0$ *when* $n \to \infty$ *and* $\Theta \to 0^+$.

Proof. We shall study the convergence in terms of the distributions $U_{ErlB}(x)$, $V_{ErlB}(x)$, $U_{ErlP}(x)$ and $V_{ErlP}(x)$, $x > 0$ of the two types of distributions.

By calculating separately three elementary limits:

$$\lim_{\substack{n\to\infty \\ \Theta\to 0^+}} (1+\Theta)^n = \lim_{\substack{n\to\infty \\ \Theta\to 0^+}} \left[(1+\Theta)^{1/\Theta}\right]^{n\Theta} = e^{\alpha},$$

$$\begin{aligned} & \lim_{\substack{n\to\infty \\ \Theta\to 0^+}} \left[1 + \Theta A(k,\lambda,x)\right]^n = \\ = \ & \lim_{\substack{n\to\infty \\ \Theta\to 0^+}} \left\{\left[1 + \Theta A(k,\lambda,x)\right]^{\frac{1}{\Theta A(k,\lambda,x)}}\right\}^{n\Theta A(k,\lambda,x)} \\ = \ & e^{\alpha A(k,\lambda,x)} \end{aligned}$$

and

$$\lim_{\substack{n\to\infty \\ \Theta\to 0^+}} \left[1 + \Theta\left(1 - A(k,\lambda,x)\right)\right]^n =$$

$$= \lim_{\substack{n\to\infty \\ \Theta\to 0^+}} \left\{ \left[1+\Theta\left(1-A(k,\lambda,x)\right)\right]^{\frac{1}{\Theta(1-A(k,\lambda,x))}} \right\}^{n\Theta(1-A(k,\lambda,x))}$$

$$= e^{\alpha(1-A(k,\lambda,x))}, \text{where } A(k,\lambda,x) = e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!},$$

we obtain:

$$\lim_{\substack{n\to\infty \\ \Theta\to 0^+}} U_{ErlB}(x) = \lim_{\substack{n\to\infty \\ \Theta\to 0^+}} \frac{\left(1+\Theta-\Theta e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)^n - 1}{(1+\Theta)^n - 1}$$

$$= \frac{e^{\alpha\left(1-e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)} - 1}{e^{\alpha}-1} = U_{ErlP}(x)$$

and

$$\lim_{\substack{n\to\infty \\ \Theta\to 0^+}} V_{ErlB}(x) = \lim_{\substack{n\to\infty \\ \Theta\to 0^+}} \frac{(1+\Theta)^n - \left(1+\Theta e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)^n}{(1+\Theta)^n - 1}$$

$$= \frac{e^{\alpha} - e^{\alpha e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}}}{e^{\alpha}-1} = V_{ErlP}(x).$$

□

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