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On existential expressibility of formulas in the simplest non-trivial super-intuitionistic propositional logic

Andrei Rusu, Elena Rusu

Abstract

We consider the well-known 3-valued extension of the intuitionistic propositional logic [1] and examine the conditions for o system of formulas to be complete with respect to existential expressibility of formulas considered earlier by A. V. Kuznetsov [2]. It was established that there exists a relative simple algorithm to determine whether a system of formulas is complete relative to existential expressibility of formulas in the 3-valued extension of the intuitionistic propositional logic.

Keywords: intuitionistic logic, existential expressibility, super-intuitionistic logic.

1 Introduction

In 1921 E. Post analysed the possibility get a formula from other formulas by means of superpositions [3, 4] and proved that there are a numerable collection of closed with respect to superpositions classes of boolean functions, among which only 5 of them are maximal with respect to inclusion. A. V. Kuznetsov have generalized the notion of superposion of functions to the case of formulas and put into consideration the notion of parametric expressibility as well as existential expressibility of a formula via a system of formulas in a given logic [2] and proved there finitely many precomplete with respect to parametric expressibility classes of formulas in the general 2-valued and 3-valued logics. It was stated in [2] that together with parametric expressibility it is also interesting to investigate the existential expressibility of formulas. The main result of the present paper is the theorem that states that there is an algorithm which allows to determine whether any formula of the simplest non-trivial super-intuitionistic logic L could be existentially expressible via a given system of formulas Σ in L.

2 Definitions and notations

Intuitionistic propositional logic Int **[5].** The calculus of the propositional intuitionistic logic Int is based on formulas built as usual from propositional variables $p, q, r, p_1, q_i, r_j, \ldots$, logical connectives $\&, \vee, \supset, \neg$ and auxiliary symbols of left and right parentheses (and). Axioms of Int are the formulas: $p \supset (q \supset p)$, $(p \supset q) \supset ((p \supset (q \supset r)) \supset (p \supset r)$, $p \supset (q \supset (p \,\&\, q))$, $p \supset (p \vee q)$, $p \supset (q \vee p)$, $(p \vee q) \supset p$, $(p \vee q) \supset q$, $(p \supset r) \supset ((q \supset r) \supset ((p \vee q) \supset r))$, $(p \supset q) \supset ((p \supset \neg q) \supset \neg p)$, $\neg p \supset (p \supset q)$. and the well-known rules of inference: *modus ponens*, and *substitution*. The intuitionistic logic Int of the above calculus is defined as usual as the set of formulas deductible in that calculus.

Any set of formulas L containing Int and closed with respect to the rules of inference is said to be an *extention of* Int, also being known as *super-intuitionistic logic* or *intermediate logic* [6]. We consider the super-intuitionistic logic L3 of the second slice defined by two additional axioms [6]:

$$Z = (p \supset q) \vee (q \supset p),$$

$$P_2 = ((r \supset [((q \supset p) \supset q) \supset q]) \supset r) \supset r$$

Existential expressibility [2]. Suppose in the logic L we can define the equivalence of two formulas. The formula F is said to be *(expricitly) expressible* via a system of formulas Σ in the logic L if F can be obtained from variables and formulas Σ using two rules: a) the rule of weak substitution, which allows to pass from two formulas, say A and B to the result of substitution of one of them in another in place of any variable $\frac{A,B}{A[B]}$ (where we denote by $A[B]$ the thought substitution); b) if we already get formula A and we know A is equivalent in L to B, then we have also formula B.

The formula F is said to be *existentially expressible* in the logic L via the system of formulas Σ if there exists variables $q_1, \ldots, q_s, q$ not occurring in F, formulas $D_1, \ldots, D_s$ and formulas $B_1, \ldots, B_m$ and $C_1, \ldots, C_m$ such that $B_{j1}, \ldots, B_{jm}$ and $C_{j1}, \ldots, C_{jm}$, $j = 1, \ldots, k$, are explicitly expressible in L via formulas of Σ and the following first-order formulas are true:

$$(F = q) \Longrightarrow (\bigvee_{j=1}^{k} \bigwedge_{i=1}^{m} (B_{ji} = C_{ji}))[q_1/D_1] \ldots [q_s/D_s],$$

$$(\bigvee_{j=1}^{k} \bigwedge_{i=1}^{m} (B_{ji} = C_{ji})) \Longrightarrow (F = q)$$

The system of formulas Σ is said to be *complete with respect to existential expressibility in the logic* L if any formula of the calculus of L is existentially expressible via formulas of Σ.

3 Main result

One of the main questions regarding existential expressibility of formulas is whether there is an algorithm for detecting in the given logic L able to detct the completeness with respect to existential expressibility of classes of formulas in L.

Theorem 1. *There is an algorithm for which is able to detect whether a given system formulas Σ is complete with respect to existential expressibility of formulas in the simplest three-valued extension of the intuitionistic logic.*

4 Conclusion

This is the first step in establishing the conditions for an arbitrary system of formulas Σ to be complete with respect to existential expressibility in the intuitionistic logic of propositions.

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Andrei Rusu[1,2], Elena Rusu[3]

[1]Ovidius University of Constanţa
Email: agrusu@univ-ovidius.ro

[2]Information Society Development Institute
Email: andrei.rusu@idsi.md

[3]Technical University of Moldova
Email: elena.rusu@mate.utm.md